An Empirical Correlation for Velocity Distribution of Turbulent Fluid Flow

B. F. RUTH and H. H. YANG

Lamar State College of Technology, Beaumont, Texas

PRANDTL, VON KARMAN, AND NIKURADSE EQUATIONS

The velocity distribution of a fluid flowing in a circular pipe has long been recognized as a problem of fundamental importance in mass and heat transfer. For the turbulent flow of an incompressible fluid, Prandtl (*3*) postulated that the shearing stress within the fully established turbulent core in a smooth pipe could be expressed by the following equation:

$$\tau = \rho l^2 \left(\frac{\partial u_m}{\partial y}\right)^2 \quad (1)$$

where l is a small increment in distance from the pipe wall and is usually called the *mixing length*. The total shearing stress of the fluid will equal the sum of apparent viscous and turbulent shearing stresses, as shown by Equation (2).

$$\tau = \mu \frac{du_m}{dy} + \rho l^2 \left(\frac{\partial u_m}{\partial y}\right)^2 \quad (2)$$

Since the viscous shearing stress is of negligible magnitude beyond the laminar boundary layer, Equation (2) may be simplified to include the turbulent shearing stress only and becomes

$$\sqrt{\frac{\tau}{\rho}} = l \frac{\partial u_m}{\partial y} \quad (3)$$

The term $\sqrt{\tau/\rho}$ has a dimension of velocity and is called the *shear velocity*, u_*. For further derivation, Prandtl made the following assumption for the vicinity of pipe wall:

$$l = ky \quad (4)$$

Therefore

$$u_* = ky \frac{\partial u}{\partial y} = ky \frac{du}{dy} \quad (5)$$

Upon integration, Equation (5) becomes

$$\frac{u_s - u}{u_*} = \frac{1}{k} \ln \frac{r}{y} \quad (6)$$

which is known as the *Prandtl equation for velocity distribution of turbulent fluid in smooth pipes*. The constant k is usually taken as 0.4 according to the experimental data of Nikuradse (*2*) for the turbulent region and 0.417 for the boundary region. For the turbulent core, therefore, Equation (6) becomes

$$\frac{u_s - u}{u_*} = 5.75 \log \frac{r}{y} \quad (6a)$$

Instead of Equation (4), Von Karman proposed to use the following expression to define the Prandtl mixing length:

$$l = k \frac{du}{dy} \bigg/ \frac{d^2u}{dy^2} \quad (7)$$

This leads to the derivation of Equation (8), which also describes the velocity distribution of a turbulent core.

$$\frac{u_s - u}{u_*} = -\frac{1}{k}\left[\sqrt{1 - \frac{y}{r}} + \ln\left(1 - \sqrt{1 - \frac{y}{r}}\right)\right] = -\frac{1}{k}[\sqrt{x} + \ln(1 - \sqrt{x})] \quad (8)$$

where $x = 1 - y/r$, the fractional radial distance from the axis of the pipe.

It may be seen that both Equations (6) and (8) indicate that $(u_s - u)/u_*$ is a function of y/r alone. A universal velocity-distribution curve may consequently be obtained by eliminating the term of maximum velocity u_s. An accepted correlation of this kind is that given following Equation (1).

$$\frac{u}{u_*} = 5.5 + 2.5 \ln \frac{yu_*\rho}{\mu} \quad (9)$$

or more conveniently expressed as

$$u^+ = 5.5 + 2.5 \ln y^+ \quad (10)$$

To examine the validity of these equations, $(u_s - u)/u_*$ vs. y/r was plotted in Figure 1, according to Equation (6*a*). Considerable discrepancy is indicated between this curve and the shaded experimental data of Nikuradse. The Nikuradse data cover a wide range of Reynolds number from 23,300 to 3,240,000. Equation (8) by Von Karman shows even greater deviation from the experimental data when plotted in the same manner as shown in Figure 1. In the derivation of Equation (6) the shearing stress τ and consequently u_* are assumed to be constant for all values of the radial distance y, i.e., from $y = 0$ to $y = r$. This assumption is certainly not in agreement with the actual behavior of fluids in turbulent flow and may be the cause of the discrepancy in both Equations (6) and (9).

The discrepancy is minimized by applying the law of linear stress distribution. Thus, the shearing stress to a fluid in turbulent flow in a smooth pipe is assumed proportional to the distance from the pipe wall,

$$\tau_x = \tau_r x \quad (11)$$

where τ_x is the shearing stress at a distance rx from the center of the pipe,

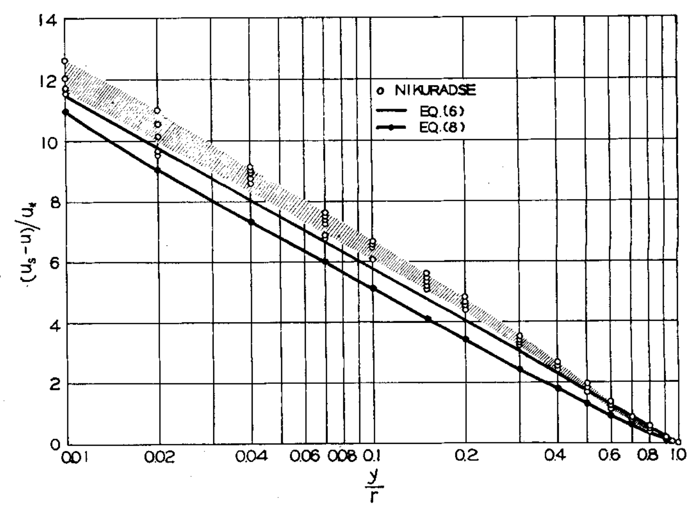


Fig. 1. Plot of $(u_S - u)/u_*$ vs. y/r.

The material presented in this paper is taken from unpublished notes of B. F. Ruth, who until his death on January 1, 1954, was professor of chemical engineering at Iowa State College. The coauthor has selected and revised the original notes and developed some of the correlations.

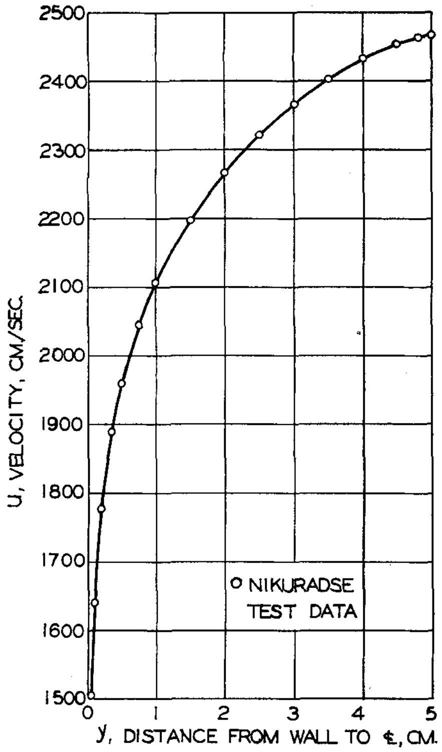


Fig. 2. Plot of u vs. y, using data of Nikuradse test No. 120.

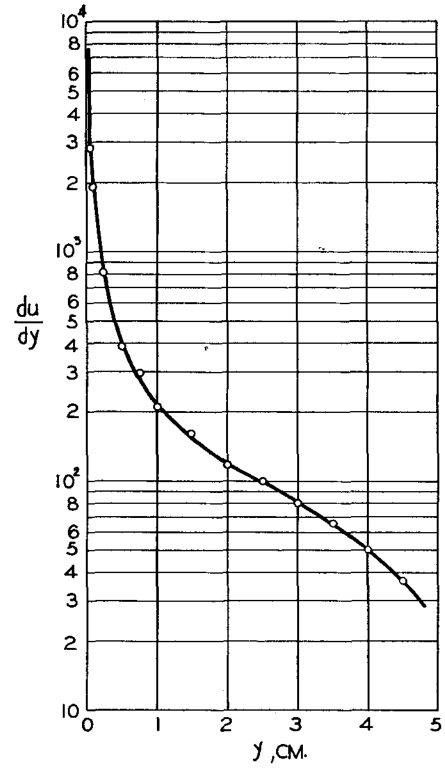


Fig. 3. Plot of du/dy vs. y, using data of Nikuradse test No. 120.

TABLE 1

EMPIRICAL CORRELATION OF PRANDTL MIXING LENGTH

x	l/r	$x^{3/2}$	$\phi = 1 - x^{3/2}$	$(1/r)(dl/d\phi)$
1.00	0	1.00	0	0.273
0.98	0.0081	0.97	0.03	0.2586
0.96	0.0154	0.941	0.059	0.25
0.93	0.0250	0.897	0.103	0.2418
0.90	0.0355	0.854	0.146	0.2175
0.85	0.0507	0.7835	0.2165	0.1932
0.80	0.0627	0.715	0.285	0.1739
0.70	0.0834	0.586	0.414	0.1482
0.60	0.0995	0.465	0.535	0.1252
0.50	0.112	0.354	0.646	0.099
0.40	0.122	0.253	0.747	0.0895
0.30	0.129	0.1642	0.8358	0.0809
0.20	0.134	0.0892	0.9108	
0.10	0.138	0.0316	0.9684	
0.04	0.1395	0.00802	0.9919	
0.02	0.1398	0.00283	9.9972	
0	0.140	0	1.00	0.9591

and τ_r that in the vicinity of the pipe wall. Substituting $\tau_r x$ for τ in Equation (3) then gives

$$\sqrt{\frac{\tau_r x}{\rho}} = u_* \sqrt{x} = l \frac{du}{dy} \quad (12)$$

or

$$\frac{du}{dy} = \frac{u_* \sqrt{x}}{l} \quad (13)$$

Since

$$y = r(1 - x)$$

$$dy = -r\, dx$$

Equation (13) may be written as

$$\frac{du}{u_*} = -\frac{\sqrt{x}\, dx}{l/r} \quad (14)$$

It can be seen that if the dimensionless ratio of l/r could be expressed as a function of x, the foregoing equation could be integrated to form a new equation representing the velocity distribution for turbulent fluid inside a smooth pipe.

EMPIRICAL CORRELATION OF VELOCITY DISTRIBUTION

To determine l/r in terms of x, a set of typical velocity data of Nikuradse test 120 was applied. The experimental data of velocity distribution were plotted in Figure 2 and differentiated graphically to yield du/dy as shown in Figure 3. Experimental values of l/r were then calculated according to Equation (13) as below:

$$\frac{du}{dy} = \frac{u_* \sqrt{x}}{l}$$

Hence,

$$\frac{l}{r} = \frac{u_* \sqrt{x}}{r\left(\frac{du}{dy}\right)} \quad (15)$$

The results were plotted against y in Figure 4.

An empirical correlation was made by Ruth, assuming l/r to be a function of $\phi = 1 - x^{3/2}$. Such a correlation was tabulated in Table 1 by use of the data of Figure 4. Values of l/r were first plotted against $1 - x^{3/2}$ in Figure 5. This curve has an exponentially decreasing slope $(1/r)(dl/d\phi)$ with respect to increasing ϕ as shown in Figure 6. This property enabled the development of an empirical equation which expresses l/r as a function of ϕ.

From Figure 6 it is seen that $(1/r)(dl/d\phi)$ has the following boundary values:

$$\frac{1}{r}\frac{dl}{d\phi} = 0.27 \text{ at } \phi = 0$$

$$\frac{1}{r}\frac{dl}{d\phi} = 0.06 \text{ at } \phi = 1$$

Hence

$$\log\left(\frac{1}{r}\frac{dl}{d\phi}\right) = \log 0.27 - 0.653\phi$$

$$\frac{1}{r}\frac{dl}{d\phi} = 0.27(10^{-0.653\phi}) = 0.27e^{-1.504\phi}$$

$$\frac{dl}{r} \cong 0.27e^{-1.5\phi}\, d\phi$$

Integrating the foregoing equation yields

$$\int_0^l \frac{dl}{r} = \int_0^\phi 0.27e^{-1.5\phi}\, d\phi$$

$$\frac{l}{r} = \frac{0.27}{1.5}(1 - e^{-1.5\phi})$$

$$\frac{l}{r} = 0.18[1 - e^{-1.5(1-x^{3/2})}] \quad (16)$$

Equation (16) is an universal expression for the Prandtl mixing length l. A graphical correlation of l/r vs. x according to this equation is presented in Figure 7.

Substituting Equation (16) into (14) gives

$$\frac{du}{u_*} = \frac{-\sqrt{x}\, dx}{0.18[1 - e^{-1.5(1-x^{3/2})}]}$$

Integration again gives

$$\int_u^{u_s} \frac{du}{u_*} = \int_x^0 \frac{-\sqrt{x}\,dx}{0.18[1 - e^{-1.5(1-x^{3/2})}]}$$

One has then the following expressions for velocity difference $(u_s - u)/u_*$ similar to Equations (6) and (8):

$$\frac{u_s - u}{u_*} = 3.08 - 5.695 \log [e^{1.5(1-x^{3/2})} - 1] \quad (17)$$

Equation (17) is considered as the final form of this empirical derivation based on Prandtl's theory of mixing length. It was analyzed and graphically presented in Figure 8 so that it might be compared with the Nikuradse experimental data of test 120. A close agreement may be seen between the two curves for the entire range of y/r. Some deviation exists in the vicinity of the pipe wall where viscous shearing stress predominates. Although the correlated curve falls within the covered area of the Nikuradse data, Equation (17) may be regarded as a satisfactory expression for the behavior of a turbulent fluid. The correlation bears slight resemblance to Equations (6) and (8) in that they all have an exponential slope of 2.5, $k = 0.4$ being taken for the latter two.

GENERALIZED CORRELATION

For a more generalized correlation of velocity distribution, the term u_s in Equation (17) has to be excluded. Nikuradse's data were analyzed with an attempt to express u_s/u_* as a function of the modified Reynolds number $ru_*\rho/\mu$. This resulted in a semilogarithmic linear plot as shown in Figure 9. The straight line may be represented by the following equation:

$$\frac{u_s}{u_*} = 6.0 + 5.7 \log \frac{ru_*\rho}{\mu} \quad (18)$$

Combination of Equations (17) and (18) then yields the following equation

$$\frac{u}{u_*} = 2.92 + 5.7 \log \left\{\frac{ru_*\rho}{\mu} [e^{1.5(1-x^{3/2})} - 1]\right\} \quad (19)$$

Thus u/u_* is no longer a function of $yu_*\rho/\mu$ or y^+ alone as is Equation (9) or (10). The velocity distribution of a turbulent fluid in terms of u/u_* is consequently not a linear function of $yu_*\rho/\mu$ on a semilogarithmic plot. This was confirmed by a careful plot of u/u_* vs. $yu_*\rho/\mu$ in Figure 10, which shows a series of wavy curves at various shearing-stress velocities. These curves propagate along the Nikuradse line with increasing Reynolds number and shearing-stress velocity. Equation (19) indicates a satisfactory agreement with the experimental data of Nikuradse as can be seen in Figure 10. Equation (9) is good but only as an average throughout the turbulent region. Its deviation from actual data is approximately ±0.5 of the u/u_* scale.

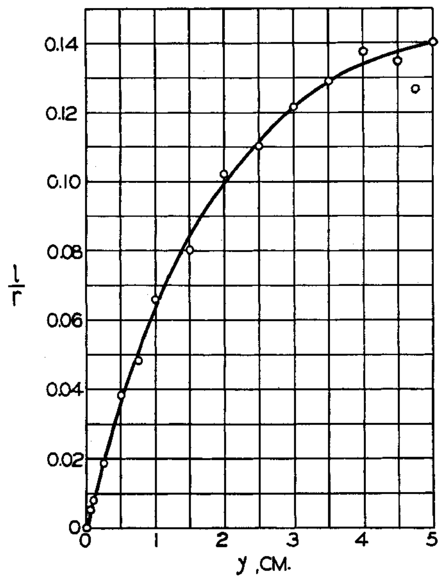


Fig. 4. Correlation of l/r vs. y, using data of Nikuradse test No. 120.

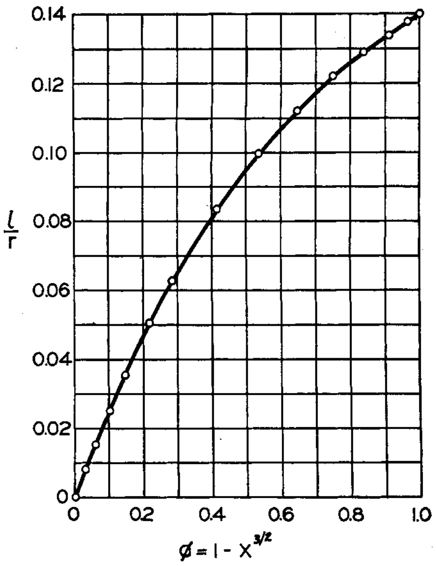


Fig. 5. Correlation of l/r vs. ϕ.

Fig. 7. Generalized correlation of l/r.

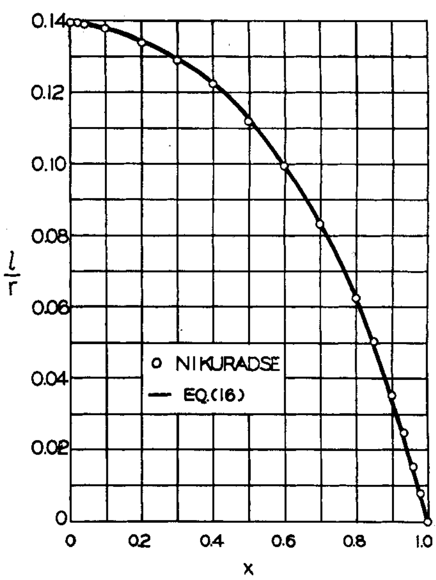
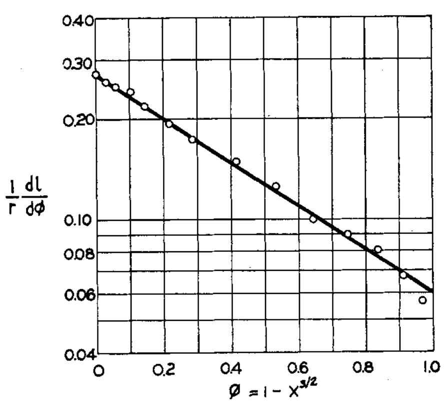


Fig. 6. Correlation of $(1/r)$ $(dl/d\phi)$ vs. ϕ.



In regard to the average velocity of fluid through a pipe, Equation (17) may be transformed to give the following:

$$\frac{u_s - u_m}{u_*} = \frac{1}{A}\int_0^A \{3.08 - 5.695 \cdot \log [e^{1.5(1-x^{3/2})} - 1]\}\, dA$$

$$= 2\int_0^1 \{3.08 - 5.695 \cdot \log [e^{1.5(1-x^{3/2})} - 1]\} x\, dx \quad (20)$$

Numerical integration of Equation (2) and examination of fifty Nikuradse tests on velocity distribution showed that

$$\frac{u_s - u_m}{u_*} = 4.07 \quad (21)$$

When u_s/u_* is substituted with Equation (18), Equation (21) becomes

$$\left(6.0 + 5.7 \log \frac{ru_*\rho}{\mu}\right)\left(1 - \frac{u_m}{u_s}\right) = 4.07$$

$$\frac{u_m}{u_s} = 1 - \frac{4.07}{6.0 + 5.7 \log \frac{ru_*\rho}{\mu}} \quad (22)$$

Although u_m/u_s could be expressed as a function of the modified Reynolds number given in Equation (22), the correlation is not so conveniently applicable as that of Senecal and Rothfus (4). Equation (22) indicates values of u_m/u_s ranging from 0.819 at $ru_*\rho/\mu = 1{,}080$

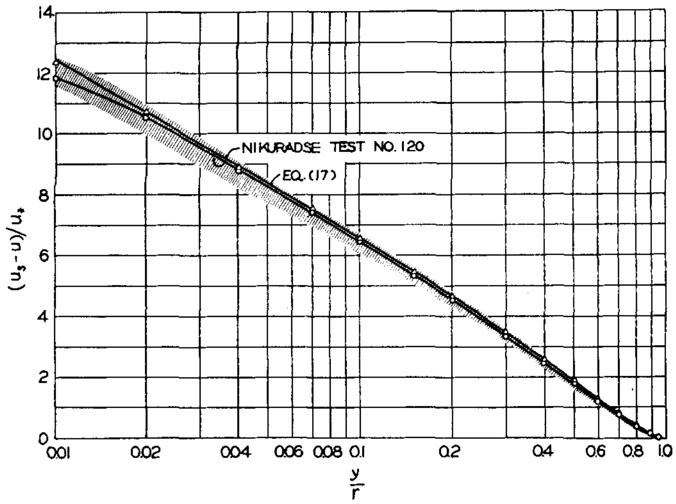


Fig. 8. Plot of $(u_S - u)/u_*$ vs. y/r.

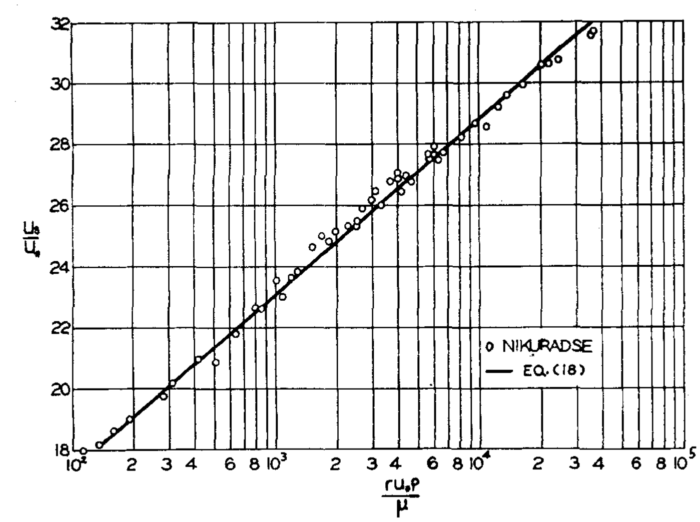


Fig. 9. Plot of u_S/u_* vs. $ru_*\rho/\mu$.

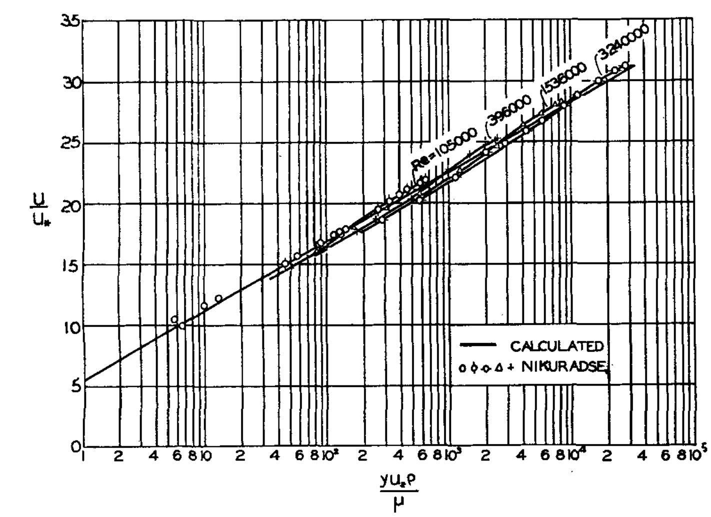


Fig. 10. Velocity distribution of turbulent fluid.

to 0.8725 at 55,900, which are in excellent agreement with the experimental values.

CONCLUSION

The foregoing empirical correlation of Ruth has established Equation (19) as a universal expression for fully developed turbulent flow in a smooth pipe. The equation shows that the velocity ratio u/u_* could be correlated against $ru_*\rho/\mu$ with the fractional radial distance x as a parameter. A plot of this equation would show a characteristic series of pulsating curves for fully developed turbulance similar to those shown in Figure 10. A generalized empirical equation for the Prandtl mixing length was also derived. Equation (16) shows the relationship between the ratio of l/r and the fractional radial distance x. The empirical correlation made possible a generalized expression for the ratio of average velocity of a turbulent fluid to its maximum velocity in terms of the modified Reynolds number $ru_*\rho/\mu$, which is given in Equation (22).

It must be stressed that the mixing-length correlation is valid only for Reynolds numbers over 100,000. This presumably restricts the final correlation to the same extent, i.e., fully developed turbulence.

NOTATION

A = internal transverse area of pipe, sq. cm.
k = proportionality constant
l = Prandtl mixing length, cm.
r = inside radius of pipe, cm.
R_e = Reynolds number = $(Du_m\rho)/\mu$
u = velocity of fluid, cm./sec.
u_m = mean velocity of fluid, cm./sec.
u_s = maximum velocity of fluid, cm./sec.
u_* = shearing-stress velocity of fluid, cm./sec. = $\sqrt{\tau/\rho}$
x = fractional radius inside pipe = $1 - y/r$
y = radial distance from pipe wall = $r(1 - x)$, cm.
μ = viscosity of fluid, g./(cm.)(sec.)
ρ = density of fluid, g./cc.
τ = total shearing stress of fluid, g. force/sq. cm.
τ_r = local shearing stress of fluid at pipe wall, g. force/sq. cm.
τ_x = local shearing stress of fluid at a fractional distance x to center of pipe
ϕ = $1 - x^{3/2}$

LITERATURE CITED

1. Martinelli, R. C., *Trans. Am. Soc. Mech. Engrs.*, **69**, 947 (1947).
2. Nikuradse, J., *V.D.I. Forschungsheft* 356 (1932).
3. Prandtl, Ludwig, "Essentials of Fluid Dynamics" [original text in German, translated into English by W. M. Deans], Hafner Publishing Company, New York (1952).
4. Senecal, V. E., and R. R. Rothfus, *Chem. Eng. Progr.*, 49, 10, 533 (1953).